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DIGITAL COMPUTER SIMULATION OF A THERMAL REGENERATOR

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Abstract—Much recent work has been devoted to the simulation of thermal regenerators, in particular, Cowper Stoves. A numerical step-by-step procedure to solve the differential equations describing the regenerative heat exchanger is set out. This uses the trapezoidal difference representation of the equations. This method has been programmed for a digital computer and a comparison made with another independent method of calculation. The two techniques yield identical solutions to the equations. It is necessary to decide upon the minimum number of steps in the space variable which require to be incorporated into the computer simulation for an accurate solution of the equations, especially in an analogue representation where step length cannot easily be altered. The factors governing the truncation errors are therefore discussed and it is shown that the larger the Hausen factor K/K_0, the fewer the steps necessary for accurate solution.

NOMENCLATURE

A, regenerator heating surface area [ft², cm²];

a, b, c, constants defined by gas composition which are used in the expression relating gas specific heat to temperature;

C, specific heat of heat storing matrix [Btu/lb degF, cal/g degC];

D, thermal diffusivity of chequerwork [ft²/h, cm²/s].

$\bar{h}$, overall heat-transfer coefficient [Btu/ft² h degF, cal/cm² s degC];

h, surface heat-transfer coefficient [Btu/ft² h degF, cal/cm² s degC];

L, length of regenerator [ft, cm];

M, mass of heat storing matrix [lb, g];

m, number of steps of integration in distance direction;

P, length of period [h, s];

p, number of steps of integration in time;

S, specific heat of gas [Btu/lb degF, cal/g degC];

t, gas temperature [degF, degC];

T, mean solid temperature [degF, degC];

W, flow rate of gas [lb/h, g/s];

y, distance from regenerator entrance [ft, cm];

K/K_0, Hausen ratio defining the extent of the non-linear temperature changes in the regenerator;

Greek symbols

γ, parameter with dimensions of specific heat [Btu/lb degF, cal/g degC];

ξ, dimensionless length;

η, dimensionless time;

θ, time [h, s];

λ, thermal conductivity of chequerwork [Btu/ft h degF, cal/cm s degC];

Λ, reduced length;

Π, reduced time;

$\Phi(n)$, pseudo-thermal ratio at the end of the n-th cycle;

η_{REG}, thermal ratio (sometimes called thermal recovery);

Δ, wall semi-thickness [ft, cm];

φ, a correction applied to account for the inversion of the parabolic solid temperature profile at the regenerator reversals.

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Subscripts

h, i, hot inlet;
c, i, cold inlet;
h, o, m, chronological mean hot outlet;
c, o, m, chronological mean cold outlet;
c, o, f, final cold outlet;
r, refers to position in y-direction;
S, refers to position in time;
0, refers to the regenerator entrance and to the beginning of a period;
m, refers to the regenerator exit;
p, refers to the end of a period;
q, refers to surface of chequerwork.

Superscripts

$'$, refers to heating period;
$''$, refers to cooling period.

INTRODUCTION

THE THERMAL regenerator consists of a heat storing matrix of solid material, often called "chequerwork". Heat is transferred between the matrix and the fluid, usually a gas, which passes through the channels in the matrix.

The cycle of operation consists of a heating period and a cooling period. In the heating period, heat is transferred to the chequerwork from the gas of specified entrance temperature and specified flow rate. At the end of the heating period, a reversal occurs at which the flow of hot gas is shut off. Then, during the cooling period, a gas of lower entrance temperature and specified flow rate passes through the channels of the chequerwork, in the opposite direction of flow to that in the heating period. Throughout the cooling period, heat is transferred from the chequerwork to the gas. The cooling period is concluded by another reversal and the next heating period commences.

The differential equations used here to describe regenerator performance are based on the following assumptions:

(i) the effect of the reversals can be neglected, that is the rapid gas temperature transients which are associated with the residual gas in the regenerator being replaced by the gas flowing in the opposite direction at the reversal can be ignored.
(ii) the entrance gas temperatures in both periods remain constant.
(iii) the mass flow rates of the heating and cooling gases do not vary throughout each period.
(iv) heat transfer between gas and solid can be represented in terms of an overall heat-transfer coefficient relating gas temperature to mean solid temperature. Further the rate of heat transfer in the chequerwork at any height is represented by the time variation of the mean solid temperature. This is discussed later.
(v) the heat capacity of the gas in the channels of the matrix at any instant is small relative to the heat capacity of the chequerwork, and therefore can be neglected.
(vi) the heat-transfer coefficients and the thermal properties of the heat storing mass and the gas do not vary throughout a period and are identical at all parts of the regenerator in that period.
(vii) longitudinal thermal conductivity is neglected.

Although these assumptions are not strictly true for the operation of many types of regenerator, the method of solving the differential equations described here can be extended to cover those features which might be included in a more sophisticated model of a regenerator.

Several techniques have been developed to solve the hyperbolic differential equations describing the performance of the regenerative heat exchanger. Hausen [1, 2], Anzelius [3] and Nusselt [4] were early contributors in the 1930's and Iliffe [5] presented an alternative method of solution of the equations in 1948.

There have been two different approaches to this problem. The first is called the closed type where the equations are solved directly for the thermal equilibrium conditions, that is when the solid temperature distribution in the matrix is identical at the end of successive cycles. Typical of this method is Hausen's eigenfunction method [1] and Nahavandi and Weinstein's Laplace transform method [10].

The second method, of which the procedure described here is an example, is the simulation type, where in effect the mathematical model is cycled to equilibrium. This simulation type of approach can be further classified into two other

types. In Hausen's heat pole method [2] and Iliffe's Bessel function method [5], use is made of Green's theorem in order to integrate chronologically, whereas in the method presented here and in the paper by Lambertson [11], numerical methods of integrating the differential equations are applied.

The present paper therefore describes a numerical technique to solve the equations by an elementary step-by-step procedure. This method has been programmed for the Ferranti "Pegasus" computer. This programme is a simulation of the performance of the regenerator.

The accuracy of the method has been checked against another independent method [5] for the solution of the differential equations based on the same assumptions.

THE DIFFERENTIAL EQUATIONS

The partial differential equations considered are of the hyperbolic type and are

$$\bar{h}A\,(T - t) = WSL\,\frac{\partial t}{\partial y} \tag{1}$$

and

$$\bar{h}A\,(t - T) = MC\,\frac{\partial T}{\partial \theta}. \tag{2}$$

The heat-transfer coefficient $\bar{h}$ relates gas and mean solid temperatures and is connected to the surface heat-transfer coefficient h by the relation

$$h\,(T_g - t) = \bar{h}\,(T - t).$$

This has been discussed by Hausen [7], Willmott [6] and Butterfield [12]. It is useful to consider a generalized wall semi-thickness, Δ, of the chequerwork material around the channels in the regenerator matrix. A further relation between h and $\bar{h}$ can be obtained, namely

$$\frac{1}{\bar{h}} = \frac{1}{h} + \frac{\Delta}{3\lambda}\varphi,$$

where φ is a correction applied to account for the inversion of the parabolic temperature profile in the walls of the matrix at the regenerator reversals. Hausen [7] has obtained an analytical relationship between φ and the expression

$$\frac{D\Delta^2}{6}\left(\frac{1}{P'} + \frac{1}{P''}\right)$$

for the "plain wall" case whilst Butterfield has computed by numerical methods the corresponding values of φ as a function of the above expression for circular cylindrical and square cylindrical channelled regenerator matrices.

The equations (1) and (2) can be simplified by introduction of the dimensionless parameters

$$\xi = \frac{\bar{h}A}{WSL}y \quad \text{and} \quad \eta = \frac{\bar{h}A}{MC}\theta.$$

The equations (1) and (2) then become

$$\frac{\partial t}{\partial \xi} = T - t \tag{3}$$

and

$$\frac{\partial T}{\partial \eta} = t - T. \tag{4}$$

These parameters give rise to the dimensionless groups, called by Hausen [2], "reduzierte Länge" (reduced length) and "reduzierte Periodendauer" (reduced period).

When $y = L$ and $\theta = P$, then

$$\Lambda = \frac{\bar{h}A}{WS} = \text{reduced length}$$

and

$$\Pi = \frac{\bar{h}AP}{MC} = \text{reduced period}$$

where P = length of period [h, s].

BOUNDARY CONDITIONS

There are two boundary conditions. Firstly, the entrance gas temperatures in both heating and cooling periods are constant. Secondly, the solid temperatures at the end of a heating/cooling period are the same as those at the beginning of the succeeding cooling/heating period. In order to include the counterflow operation of the regenerator, i.e. that the gases flow in opposite directions in succeeding periods, this boundary condition is expressed as

$$T'(0, y) = T(P, L - y)$$

where the prime refers to the succeeding period. For the first cycle considered, $T(0, y)$ is defined arbitrarily for $0 \leqslant y \leqslant L$.

METHOD OF SOLUTION

The methods of calculation such as those proposed by Hausen (the Warmepol Method) [2] and Iliffe (the Bessel Function Method) [5] were developed before the advent of digital computers. They were thus especially suitable for hand calculations using either a desk calculating machine or a large slide rule together with books of tabulated functions.

In the present method, which is particularly suitable for a digital computer programme, both equations are expressed in difference form, and integrated using the trapezoidal method. The resulting difference equations when solved yield the values of the gas and solid temperatures at equally spaced distances $\Delta\xi$ down the regenerator, and at equally spaced intervals of time $\Delta\eta$. This can be represented conveniently by the lattice as in Fig. 1. The suffixes r and S refer to distance and time respectively.

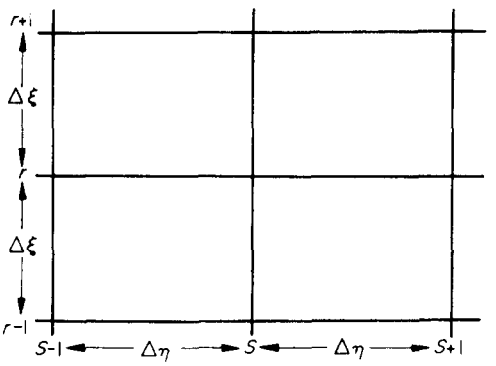

FIG. 1.

Employing the trapezoidal method, the difference forms of equations (3) and (4) are

$$t_{r+1,S} = t_{r,S} + \frac{\Delta\xi}{2}\left\{\left(\frac{\partial t}{\partial \xi}\right)_{r+1,S} + \left(\frac{\partial t}{\partial \xi}\right)_{r,S}\right\} \quad (5)$$

and

$$T_{r,S+1} = T_{r,S} + \frac{\Delta\eta}{2}\left\{\left(\frac{\partial T}{\partial \eta}\right)_{r,S+1} + \left(\frac{\partial T}{\partial \eta}\right)_{r,S}\right\} \quad (6)$$

The truncation errors associated with these difference representations are respectively

$$C_\xi = -\left(\frac{\Delta\xi}{12}\right)^3 \left(\frac{\partial^3 t}{\partial \xi^3}\right)_{r,S} + \cdots$$

and

$$C_\eta = -\left(\frac{\Delta\eta}{12}\right)^3 \left(\frac{\partial^3 T}{\partial \eta^3}\right)_{r,S} + \cdots$$

If $\frac{1}{2}\Delta\xi$ is denoted by α and $\frac{1}{2}\Delta\eta$ by β and, further, the following general substitutions are made:

$$\left(\frac{\partial t}{\partial \xi}\right)_{r,S} = T_{r,S} - t_{r,S}$$

$$\left(\frac{\partial T}{\partial \eta}\right)_{r,S} = t_{r,S} - T_{r,S},$$

then it follows that equation (5) can be reduced to the form

$$t_{r+1,S} = \left(\frac{1-\alpha}{1+\alpha}\right) t_{r,S} + \frac{\alpha}{(1+\alpha)}(T_{r+1,S} + T_{r,S}). \quad (7)$$

Similarly, equation (6) becomes

$$T_{r,S+1} = \left(\frac{1-\beta}{1+\beta}\right) T_{r,S} + \frac{\beta}{(1+\beta)}(t_{r,S+1} + t_{r,S}). \quad (8)$$

THE INTEGRATION PROCEDURE

In order to begin the integration, it is necessary to establish certain initial conditions. These are defined by the boundary conditions, namely

(i) $t_{0,S}$ = constant for all S in the particular period.

(ii) $T_{r,0}$ is defined for all r arbitrarily at the beginning of the first cycle and then by the boundary condition $T'(0, y) = T(P, L - y)$ at the beginning of the successive periods.

At the beginning of a period, therefore, the initial solid temperatures $T_{r,0}$ are known for all $r = 0, 1, 2 \ldots\ldots m$, (where $m\Delta\xi = \Lambda$) together with the inlet gas temperature $t_{0,0}$. The gas temperatures at the beginning of the period down the length of the regenerator are therefore calculated, using the difference equation (7),

$$t_{r+1,0} = \left(\frac{1-\alpha}{1+\alpha}\right) t_{r,0} + \frac{\alpha}{(1+\alpha)}(T_{r+1,0} + T_{r,0})$$

for $r = 1, 2, \ldots\ldots m$.

This assumes that the disturbance to the system at the regenerator reversals can be ignored.

The procedure now is to evaluate $T_{r,1}$ for $r = 0, 1 \ldots\ldots m$ and $t_{r,1}$ for $r = 1, 2 \ldots m$,

given the values of $T_{r,0}$ and $t_{r,0}$ for $r = 0, 1 \ldots . m$ and $t_{0,1}$, the constant inlet gas temperature.

It is now convenient to rewrite equations (7) and (8) in the form

$$t_{r+1,S} = A_1\, t_{r,S} + A_2\,(T_{r+1,S} + T_{r,S}) \quad (7a)$$

$$T_{r,S+1} = B_1\, T_{r,S} + B_2\,(t_{r,S+1} + t_{r,S}) \quad (7b)$$

with

$$A_1 = \frac{1-\alpha}{1+\alpha}, \qquad A_2 = \frac{\alpha}{1+\alpha},$$

$$B_1 = \frac{1-\beta}{1+\beta}, \qquad B_2 = \frac{\beta}{1+\beta}.$$

Equation (7b) involves prior knowledge of $t_{r,S+1}$. But

$$t_{r,S+1} = A_1\, t_{r-1,S+1} + A_2\,(T_{r,S+1} + T_{r-1,S+1}).$$

The following form of the equation can thus be derived:

$$T_{r,S+1} = B_1\, T_{r,S} + B_2\,\{A_1\, t_{r-1,S+1} + A_2\,(T_{r,S+1} + T_{r-1,S+1}) + t_{r,S}\}.$$

This becomes

$$T_{r,S+1} = \frac{B_1}{(1 - A_2\, B_2)}\, T_{r,S} + \frac{B_2}{(1 - A_2\, B_2)}\, t_{r,S} + \frac{A_2\, B_2}{(1 - A_2\, B_2)}\, T_{r-1,S+1} + \frac{A_1\, B_2}{(1 - A_2\, B_2)}\, t_{r-1,S+1}$$

or

$$T_{r,S+1} = K_1\, T_{r,S} + K_2\, t_{r,S} + K_3\, T_{r-1,S+1} + K_4\, t_{r-1,S+1}, \quad (9)$$

where

$$K_1 = \frac{B_1}{1 - A_2\, B_2}, \qquad K_2 = \frac{B_2}{1 - A_2\, B_2},$$

$$K_3 = \frac{A_2\, B_2}{1 - A_2\, B_2}, \qquad K_4 = \frac{A_1\, B_2}{1 - A_2\, B_2}.$$

It is now possible to evaluate $T_{1,1}$, that is

$$T_{1,1} = K_1\, T_{1,0} + K_2\, t_{1,0} + K_3\, T_{0,1} + K_4\, t_{0,1}.$$

From this, $t_{1,1}$ can be calculated using equation (7a), namely

$$t_{1,1} = A_1\, t_{0,1} + A_2\,(T_{1,1} + T_{0,1}).$$

This procedure is repeated continually up the whole length of the regenerator. In general terms, this can be represented diagrammatically as in Fig. 2.

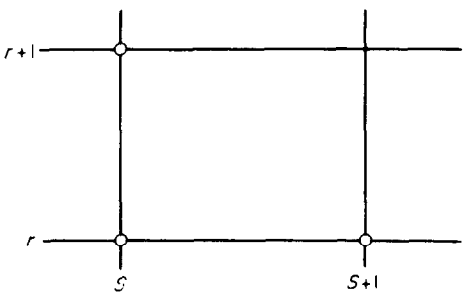

FIG. 2.

At any stage of the integrating procedure, the values of T and t are known at the points indicated, namely (r, S), $(r, S+1)$ and $(r+1, S)$. Using equation (9) the value of T at $(r+1, S+1)$ is obtained explicitly. Then, using equation (7a) the value of t at $(r+1, S+1)$ is also calculated explicitly.

The integration is continued over the whole period, that is for $S = 0, 1, 2 \ldots . p$ where $p\Delta\eta = \Pi$. At this stage the counterflow reversal condition, namely $T''(0, y'') = T'(P, L - y'')$, is applied and integration over the succeeding period continues.

One cycle of regenerator operation is comprised of one heating/cooling period followed by one cooling/heating period. Each period is defined by its reduced length Λ, its reduced period Π and its constant inlet gas temperature $t_{0,S}$.

After a large number of successive cycles, the solution of the difference equations becomes independent of the initial arbitrary solid temperatures. This state is known as "cyclic equilibrium", and it is to this condition that the solution of the equations converges. It is this condition which was dealt with by Hausen and Iliffe in their treatments of the equations.

When cyclic equilibrium is reached, the thermal ratio η_{REG} for both periods is calculated. These ratios are defined by expressions

$$\eta'_{\mathrm{REG}} = \frac{t_{h,i} - t_{h,o,m}}{t_{h,i} - t_{c,i}} \text{ for the heating period}$$

and

$$\eta''_{\mathrm{REG}} = \frac{t_{c,o,m} - t_{c,i}}{t_{h,i} - t_{c,i}} \text{ for the cooling period.}$$

In order to calculate the time mean exit gas temperatures, Gregory's formula [8] is applied. That is

$$\text{time mean exit gas temperature} = \frac{1}{\Pi}\int_0^{\Pi} t_m(\eta)\,\mathrm{d}\eta$$

and is represented by

$$\frac{1}{p}\left[\frac{1}{2}t_{m,0} + \sum_{S=1}^{p-1} t_{m,S} + \frac{1}{2}t_{m,p} - \frac{1}{12}(\nabla t_{m,p} - \Delta t_{m,0}) - \frac{1}{24}(\nabla^2 t_{m,p} + \Delta^2 t_{m,0}) - \frac{19}{720}(\nabla^3 t_{m,p} - \Delta^3 t_{m,0})\right]$$

where Δ and ∇ are used here as the forward and backward differences respectively.

Iliffe [5] demonstrated that when the water equivalent flow per period, *WSP*, is the same for both the hot entrance gas and the cold entrance gas, then

$$\eta'_{\mathrm{REG}} = \eta''_{\mathrm{REG}}.$$

This condition is satisfied when $\Pi'/\Lambda' = \Pi''/\Lambda''$ where Π' and Λ' refer to the heating period and Π'' and Λ'' refer to the cooling period. Under these limiting conditions, it is often convenient to determine whether dynamic equilibrium has been reached by comparing η'_{REG} and η''_{REG} at the end of each cycle.

For the condition $\Pi'/\Lambda' \neq \Pi''/\Lambda''$, it is convenient to compare η'_{REG} (or η''_{REG}) at the end of successive cycles, in order to determine whether the simulation has converged to equilibrium.

In the programme for the Ferranti "Pegasus" computer, in order that the chronological mean exit gas temperatures need only be calculated for the equilibrium cycle, a pseudo-thermal ratio is calculated at the end of each cooling period. This is defined by

$$\Phi(n) = \frac{t_{c,o,f} - t_{c,i}}{t_{h,i} - t_{c,i}},$$

When $\Phi(n) - \Phi(n-1)$ is less than a pre-specified small number, the $(n+1)$th cycle is regarded as the equilibrium cycle.

THE ACCURACY OF THE METHOD

In order to assess the accuracy of the numerical procedure, a comparison is presented between results calculated by a computer programme using the method, and those set out by Iliffe [5] in his paper. The effect of the truncation errors is considered later. In this comparison the availability of high speed computing enabled small step lengths in distance ($\Delta\xi = 0{\cdot}3$ to $0{\cdot}9$) and small time steps ($\Delta\eta = 0{\cdot}1$ to $0{\cdot}25$) to be taken without inconvenience.

Table 1. *Values of* $\eta'_{\mathrm{REG}} = \eta''_{\mathrm{REG}}$
$\Lambda'/\Lambda'' = \Pi'/\Pi'' = 2$

Λ''	$\Pi'' = 6$		$\Pi'' = 12$	
	Iliffe	Willmott	Iliffe	Willmott
3	0·453	0·4531	0·249	0·2493
6	0·709	0·7086	0·487	0·4865
12	0·865	0·8647	0·791	0·7901
18	—	0·9122	0·885	0·8854

The author questions Iliffe's value of 0·791 for η_{REG} for $\Lambda' = \Pi' = 24$, $\Pi'' = \Lambda'' = 12$. The computed value of 0·7901 was not affected by reducing the already small step lengths $\Delta\xi$ and $\Delta\eta$ used for computing this value.

FURTHER CONSIDERATIONS OF THE INTEGRATION PROCEDURE

Upon differentiation of equation (3) with respect to η and equation (4) with respect to ξ, and application of the equality $\partial t/\partial\xi = -\partial T/\partial\eta$, equations (3) and (4) become

$$\frac{\partial^2 t}{\partial\xi\,\partial\eta} + \frac{\partial t}{\partial\xi} + \frac{\partial t}{\partial\eta} = 0 \qquad (10)$$

and

$$\frac{\partial^2 T}{\partial\xi\,\partial\eta} + \frac{\partial T}{\partial\xi} + \frac{\partial T}{\partial\eta} = 0. \qquad (11)$$

Any curve in the $\xi - \eta$ plane such that the equation

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = 0 \quad \text{or} \quad \frac{\mathrm{d}\xi}{\mathrm{d}\eta} = 0$$

is satisfied at each point will be a characteristic of the pair of equations (10) and (11). Such

curves are in fact the straight lines parallel to the ξ and η axes. The integration procedure along the ξ and η directions described in this paper is therefore a process of integration along the characteristics of the system of hyperbolic partial differential equations. This fact explains why the integration procedure is simple and explicit. As integration progresses in either the ξ or η direction, no estimation of the derivative in a direction across the direction of integration is required.

When, however, the specific heat of the gas is considered to be a function of temperature, for example, the procedure becomes much more complicated. In effect one of the two sets of characteristics is no longer just parallel to the y-axis and its geometry has to be recalculated at each step of integration. A possible procedure is described in the Appendix to this paper.

TRUNCATION ERRORS

The effect of the truncation errors C_ξ and C_η is complex. The maximum values of $\Delta\xi$ and $\Delta\eta$ permissible for convergence to a sufficiently accurate solution are functions of the values of Π and Λ for both periods being considered.

Under certain conditions, the variations of temperature with respect to both ξ and η are almost strictly linear and thus the value of the truncation errors C_ξ and C_η will be negligible, since the third order derivatives will be nearly zero. On the other hand, there are two conditions which give rise to non-linear variation of temperature with respect to ξ and η. These are:

(i) decreasing values of reduced length Λ and increasing values of reduced period Π.
(ii) increasing values of the difference between the water equivalents per period of the gas flow in the two periods of the cycle.

Under either or both of these two conditions, the non-linear variations of temperature will cause the value of the truncation errors C_ξ and C_η to increase correspondingly.

Condition (ii) is not one that is met to a serious degree in practice. However, the effect of non-linear variations of temperature due to condition (i) is a practical one. It has been described by Hausen [7] in terms of a factor K/K_0 where

$$\frac{K}{K_0} = \frac{\eta_{REG}}{1 - \eta_{REG}} \left(\frac{1}{\Lambda'} + \frac{1}{\Lambda''} \right)$$

and where $0 < K/K_0 < 1$.

The smaller the value of K/K_0, the greater the effect of both the non-linear variations of temperature and the corresponding truncation errors. It should be noted that implicit in this definition of K/K_0 is that $W'S'P' = W''S''P''$ since it is assumed that $\eta'_{REG} = \eta''_{REG}$ (see Iliffe [5], p. 367).

In his paper Hausen [7] plots the value of K/K_0 against the overall reduced length Λ_T and the overall reduced period Π_T where these are taken to be the harmonic means of the values in the two periods, that is

$$\frac{1}{\Lambda_T} = \frac{1}{2}\left(\frac{1}{\Lambda'} + \frac{1}{\Lambda''} \right) \text{ and } \frac{1}{\Pi_T} = \frac{1}{2}\left(\frac{1}{\Pi'} + \frac{1}{\Pi''} \right).$$

The present author has investigated the effect of K/K_0 upon the truncation errors, with particular reference to the maximum permissible value of $\Delta\xi$. For the purpose of discussion, the two integers m and p are defined again by $m(\Delta\xi) = \Lambda$ and $p(\Delta\eta) = \Pi$. Further $m'(\Delta\xi) = \Lambda', m''(\Delta\xi) = \Lambda'', p'(\Delta\eta) = \Pi'$ and $p''(\Delta\eta) = \Pi''$.

ILLUSTRATION CALCULATIONS

Case 1

A thermally unsymmetric case has been considered with $\Lambda' = \Pi' = 6$ and $\Lambda'' = \Pi'' = 3{\cdot}5$ ($K/K_0 = 0{\cdot}73$). The computer simulation reached equilibrium in 7 cycles from the starting condition that the initial isothermal solid temperature was 500°. The hot inlet gas temperature was 1000° with the cold inlet gas temperature 0°. In all cases $p' = 12$, $p'' = 6$.

The exit gas temperature at the end of each period in the equilibrium cycle is set out below as calculated with different values of m (the same for each period).

m	Final exit gas temperature (cooling period)	Final exit gas temperature (heating period)
4	698·37	342·11
8	694·29	344·64
16	693·29	345·25
31	693·04	345·40

This suggests m should be about 12 for an accuracy of 0·5 degrees in the final exit gas temperature. The harmonic mean reduced length is 4·421.

Case 2

$\Lambda' = 13{\cdot}7875 \quad \Lambda'' = 7{\cdot}4916$

$\Pi' = 6{\cdot}0265 \quad \Pi'' = 3{\cdot}2746 \; (K/K_0 = 0{\cdot}88)$

$p' = 24 \quad p'' = 12$

15 cycles to equilibrium

m	Final exit gas temperature (cooling period)	Final exit gas temperature (heating period)
4	642·46	357·77
8	643·80	358·23
16	644·14	358·18
31	644·22	358·16

Thus for an accuracy of 0·5 degrees, m should be about 18. The harmonic mean reduced length is 9·708.

Case 3

$\Lambda' = 24{\cdot}1702 \quad \Lambda'' = 11{\cdot}2475$

$\Pi' = 4{\cdot}8556 \quad \Pi'' = 2{\cdot}5537 \; (K/K_0 = 0{\cdot}93)$

$p' = 24 \quad p'' = 12$

m	Final exit gas temperature (cooling period)	Final exit gas temperature (heating period)	Cycles to equilibrium
4	705·04	129·90	42
8	704·60	128·04	44
16	704·46	128·21	45

It might be expected that for an accuracy of 0·5 degrees, m should be about 5. The harmonic mean reduced length is 15·3516.

It seems evident from these results, as summarized below, that the number of steps m required for a sufficiently accurate representation by the equations is not defined solely by the reduced length Λ as might be expected but more particularly by the ratio K/K_0.

Reduced length Λ_T	Hausen factor K/K_0	Number of steps for 0·5° accuracy m
4·421	0·73	12
9·708	0·88	8
15·352	0·93	5

It must be stressed that these calculations refer to the difference approximations described in this paper. Where the difference approximation is not as good then the number of steps for a required accuracy will increase. On the other hand, where a better approximation is employed, this number of steps will decrease.

Employing this trapezoidal method of integration, the minimum number of time and length steps required for the integration procedure to calculate the final exit gas temperature to an accuracy of 0·5° for the 0–1000° operating range has been estimated for a fixed reduced length $\Lambda = 10$, equal for periods of both heating and cooling, and for successive values of reduced period Π, equal in both periods. These values of Π correspond to successively decreasing values of K/K_0. It has been found in this particular case that the required number of time steps and number of length steps are nearly equal. The minimum number of length steps, $m_{\min}$, is tabulated below.

Reduced length $\Lambda = 10$ ($\Lambda' = \Lambda''$)

Reduced period Π ($\Pi' = \Pi''$)	Hausen ratio K/K_0	Thermal ratio η_{REG}	$m_{\min}$
2	0·970	0·8290	4
3·8	0·903	0·8186	6
5	0·845	0·8087	8
6	0·794	0·7987	10
7	0·738	0·7867	12
7·8	0·692	0·7757	14
8·7	0·639	0·7617	16
9·4	0·598	0·7495	18

It is interesting to note that Buker and Simcic [9] make reference to the fact that refined difference approximations to the differential equations

are more important in the accurate representation of stove performance than the introduction of refined thermal data to the model. In their digital computer representation of the equations, their value of m was 24. Such a high value can only be explained by the fact they considered a very wide range of flow rates for the stove and that the higher flow rates result in values of K/K_0 as low as 0·75 with a large value for reduced length Λ. Further, non-linearities in the variation of temperature with respect to ξ and η, even at high values of K/K_0, may be introduced by the inclusion of a gas specific heat as a function of gas temperature in the equations. Such non-linearities would of course increase the truncation errors and increase the minimum number $m_{\min}$ required for accurate solution.

THE EFFECT OF NON-LINEAR VARIATIONS OF TEMPERATURE UPON CLOSED METHODS OF SOLUTION OF THE EQUATIONS

In the method presented by Nahavandi and Weinstein, the solid temperatures $T'(\xi')$ at the end of heating period are related by an integral equation to the unknown solid temperatures at the end of the cooling period $T''(\xi'')$. Similarly the solid temperatures at the end of the cooling period are related to the unknown solid temperatures at the end of the heating period. It is then assumed that the unknown functions $T'(\xi')$ and $T''(\xi'')$ can be represented by power series expansions,

$$T'(\xi') = \sum_{n=0}^{\infty} a_n (\xi')^n$$

$$T''(\xi'') = \sum_{n=0}^{\infty} b_n (\xi'')^n.$$

By considering truncated power series, up to power N, $2N + 2$ simultaneous integral equations are solved for the coefficients a_0, a_1, $a_2 \ldots a_N$ and b_0, b_1, $b_2 \ldots b_N$. It is clear that the lower the value of Hausen K/K_0 factor (and/or the greater the difference between $W'S'P'$ and $W''S''P''$), the more terms in the power series expansions that need to be considered for accurate solution of the equations. This leads to the question of whether, for low values of K/K_0, the power series expansion is better replaced by a more rapidly convergent Chebyshev series, that is

$$T'(\xi') = \sum_{n} C_n U_n (\xi')$$

$$T''(\xi'') = \sum_{n} d_n U_n (\xi'')$$

where $U_r(x)$ is defined by

$$U_r(x) = \cos(r \cos^{-1} x).$$

In this case it is very probable that the system will be as accurately represented by a small number of simultaneous integral equations. Inspection of the analytical solution of equation (4) at the entrance to the regenerator,

$$T(\eta) = T(0)\, e^{-\eta} + t\,(1 - e^{-\eta}),$$

may suggest that a polynomial expansion might be better replaced by a series of exponentials for accurate solution of the equation by as few as possible simultaneous integral equations for low values of K/K_0.

CONCLUSIONS

The trapezoidal method of difference representation has been applied to the hyperbolic differential equations describing the temperature behaviour of the thermal regenerator. This has led to a simple and rapid numerical step-by-step procedure to calculate the gas and mean solid temperatures in a regenerator at equal intervals of time at equally spaced heights in the regenerator. This method has been programmed for the Ferranti "Pegasus" computer and has been shown to yield the same solution to the equations for the same given conditions as a different and independent method of calculation, namely that of Iliffe [5].

It has been demonstrated that the Hausen factor K/K_0 can be used to indicate the magnitude of the truncation errors associated with the difference representation for specified reduced lengths Λ' and Λ'' and reduced periods Π' and Π''. The smaller the value of K/K_0, the more steps m and p, in the distance and time directions respectively, which are required for accurate solution of the equations.

Finally, it is indicated in an appendix that the method can be extended to include the temperature dependence of the thermal properties of the chequer material and the gases, and of the heat-transfer coefficients, together

with the time dependence of the flow rate and convective heat-transfer coefficients.

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APPENDIX

The numerical step-by-step procedure described here can be extended to include the time variation of the heat-transfer coefficients and mass flow rate of the gas and the temperature dependence of the thermal properties of the gas and the material of the chequerwork and of the heat-transfer coefficients. To illustrate this, the temperature dependence of the gas specific heat will be considered.

The specific heat of a gas can be expressed as a function of gas temperature of the form

$$S(t) = a + bt + ct^{-2}$$

where

$S(t)$ = the specific heat of gas at temperature t [Btu/lb degF, cal/g degC];

t = gas temperature (°F, °C)

and a, b, c are constants determined by the gas composition.

The differential equations become

$$\frac{\partial t}{\partial y} = \frac{\bar{h}A}{W}(a + bt + ct^{-2})^{-1}(T - t)$$

$$\frac{\partial T}{\partial \theta} = \frac{\bar{h}A}{MC}(t - T).$$

Set

$$\gamma = \frac{\bar{h}A}{W}y$$

and

$$\eta = \frac{\bar{h}A}{MC}\theta.$$

It should be noted that γ has dimensions of Btu/lb degF, cal/g degC. Then the above equations become

$$\frac{\partial t}{\partial \gamma} = \frac{T - t}{a + bt + ct^{-2}}.$$

and

$$\frac{\partial T}{\partial \eta} = t - T.$$

Expressing these equations in difference form, using the trapezoidal rule, we have

$$t_{r+1,s} = t_{r,s} + \frac{\Delta\gamma}{2}\left\{\left(\frac{\partial t}{\partial \gamma}\right)_{r,s} + \left(\frac{\partial t}{\partial \gamma}\right)_{r+1,s}\right\}$$

and

$$T_{r,s+1} = T_{r,s} + \frac{\Delta\eta}{2}\left\{\left(\frac{\partial T}{\partial \eta}\right)_{r,s} + \left(\frac{\partial T}{\partial \eta}\right)_{r,s+1}\right\}$$

Upon substituting the functions t and T for the several derivatives, the equations become

$$t_{r+1,s} = t_{r,s} + \frac{\Delta\gamma}{2}\left\{\frac{T_{r,s} - t_{r,s}}{a + bt_{r,s} + ct_{r,s}^{-2}} + \frac{T_{r+1} - t_{r+1,s}}{a + bt_{r+1,s} + t_{r+1,s}^{-2}}\right\}$$

and

$$T_{r,s+1} = \frac{1-\beta}{1+\beta}T_{r,s} + \frac{\beta}{1+\beta}(t_{r,s+1} + t_{r,s})$$

where, as previously, $\beta = \frac{1}{2}\Delta\eta$.

In general, at any stage of the integration, the values of T and t will be known at the mesh points $(r, S), (r+1, S)$ and $(r, S+1)$. It is necessary to calculate the values of T and t at $(r+1, S+1)$. It is known that

$$t_{r+1,s+1} = t_{r,s+1} + \frac{\Delta\gamma}{2}\left\{\left(\frac{\partial t}{\partial\gamma}\right)_{r+1,s+1} + \left(\frac{\partial t}{\partial\gamma}\right)_{r,s+1}\right\} \quad (10)$$

But $\left(\frac{\partial t}{\partial\gamma}\right)_{r+1,s+1}$ involves $T_{r+1,s+1}$, not yet known.

However, upon the substitution of

$$T_{r+1,s+1} = \frac{1-\beta}{1+\beta}T_{r+1,s} + \frac{\beta}{1+\beta}(t_{r+1,s+1} + t_{r+1,s})$$

into the term $\left(\frac{\partial t}{\partial\gamma}\right)_{r+1,s+1}$ in the equation (10), we obtain

$$t_{r,s+1} - t_{r+1,s+1} + \frac{\Delta\gamma}{2}\left\{\left[\left(\frac{1-\beta}{1+\beta}\right)T_{r+1,s} + \frac{\beta}{1+\beta}t_{r+1,s} + \frac{1}{1+\beta}t_{r+1,s+1}\right]\Big/ (a + bt_{r+1,s+1} + ct_{r+1,s+1}^{-2}) + \left[T_{r,s+1} - t_{r,s+1}\right]\Big/ (a + bt_{r,s+1} + ct_{r,s+1}^{-2})\right\} = 0.$$

This is a non-linear equation in $t_{r+1,s+1}$ which is written as

$$g(t_{r+1,s+1}) = 0.$$

This can be solved by the Newton method, that is

$$t_{r+1,s+1}^{(n+1)} = t_{r+1,s+1}^{(n)} - g(t_{r+1,s+1}^{(n)})/g'(t_{r+1,s+1}^{(n)})$$

where $t_{r+1,s+1}^{(n)}$ is the nth iterate of $t_{r+1,s+1}$.

Having evaluated $t_{r+1,s+1}$ in this manner, it follows that $T_{r+1,s+1}$ is calculated explicitly using

$$T_{r+1,s+1} = \frac{1-\beta}{1+\beta}T_{r+1,s} + \frac{\beta}{1+\beta}(t_{r+1,s+1} + t_{r+1,s}).$$

The repeated solution of a non-linear equation, $g(t_{r+1,s+1}) = 0$, at each stage of the integration may well render this method too time-consuming. The use of Newton's method of solving non-linear equations to solve difference equations is not unknown, but it may be preferable to use a predictor–corrector method.

An example of a predictor formula is

$$t_{r+1,s+1} = t_{r-1,s+1} + 2\Delta\gamma\left(\frac{\partial t}{\partial\gamma}\right)_{r,s+1}$$

(mid-ordinate rule).

Using this predicted value of $t_{r+1,s+1}$, the derivative at $(r+1, S+1)$ is explicitly evaluated, and a corrected value of $t_{r+1,s+1}$ obtained using again the trapezoidal rule, i.e.

$$t_{r+1,s+1} = t_{r,s+1} + \frac{\Delta\gamma}{2}\left\{\left(\frac{\partial t}{\partial\gamma}\right)_{r,s+1} + \left(\frac{\partial t}{\partial\gamma}\right)_{r+1,s+1}\right\},$$

employing $T_{r,s+1}$ to form the first estimate of $(\partial t/\partial\gamma)_{r+1,s+1}$.

Using this corrected value of $t_{r+1,s+1}$, the derivative is re-evaluated at $(r+1, S+1)$ and a new corrected value of $t_{r+1,s+1}$ obtained using the trapezoidal rule. This process is iterated until $t_{r+1,s+1}$ converges.

Next, $T_{r+1,s+1}$ must be computed using the equation

$$T_{r+1,s+1} = \frac{1-\beta}{1+\beta} T_{r+1,s} + \frac{\beta}{1+\beta}(t_{r+1,s} + t_{r+1,s+1}).$$

This next new value of $T_{r+1,s+1}$ is used to re-estimate $\left(\frac{\partial t}{\partial \gamma}\right)_{r+1,s+1}$ and the procedure is iterated to convergence.

The solution of the non-linear equation $g = 0$ by Newton's method and the application of a predictor–corrector procedure both involve iterations. Future work may show that a combination of these methods will minimize the necessary computation and as a beginning it is suggested that the value of $t_{r+1,s+1}$ predicted by the mid-ordinate formula can be used as the first estimate in Newton's iteration procedure.

Résumé—Beaucoup de travaux récents ont été consacrés à la simulation des récupérateurs de chaleur, en particulier, des fourneaux Cowper. Un procédé numérique pas-à-pas de résolution des équations différentielles décrivant l'échangeur de chaleur par récupération est mis en route. Celui-ci utilise la représentation des équations par des différences trapézoïdales. Cette méthode a été programmée pour un calculateur numérique et une comparaison a été faite avec une autre méthode indépendante du calcul. Les deux techniques fournissent des solutions identiques des équations. Il est nécessaire de décider du nombre minimal de pas dans la variable spatiale que l'on a besoin d'incorporer dans la simulation par le calculateur afin d'avoir une solution précise des équations, spécialement dans une représentation analogique où la longueur du pas ne peut pas être facilement modifiée. Les facteurs gouvernant les erreurs de troncature sont donc discutées et on montre que plus le facteur de Hausen K/K_0 est grand, plus le nombre de pas nécessaires pour une solution précise est faible.

Zusammenfassung—Die Simulation thermischer Regeneratoren, insbesondere von Cowper-Öfen, wurde in letzter Zeit eingehend bearbeitet. Ein numerisches Schrittverfahren wurde zur Lösung der Differentialgleichungen der regenerativen Wärmeübertrager aufgestellt. Es benutzt die trapezförmige Differenzdarstellung der Gleichungen. Diese Methode wurde für eine Digitalrechenmaschine programmiert und es wurde eine Vergleichsrechnung nach einer anderen, unabhängigen Methode durchgeführt. Die beiden Rechenarten ergeben identische Lösungen der Gleichungen. Es ist erforderlich, über die minimale Anzahl der Schritte für die Raumvariable zu entscheiden, da diese für die genaue Lösung der Gleichungen in die Simulation durch die Rechenmaschine aufgenommen werden muss und speziell bei Analogdarstellung die Schrittlänge nur schwer geändert werden kann. Die Faktoren, welche die Fehler durch Verkleinerung der Schrittzahl bestimmen, werden deshalb diskutiert und es wird gezeigt, dass für die Lösung umso weniger Schritte nötig sind, je grösser der Hausenfaktor K/K_0 ist.

Аннотация—Многие из недавно опубликованных работ посвящены моделированию тепловых регенераторов, в частности, кауперских печей. Разработан численный метод последовательных приближений для решения дифференциальных уравнений, описывающих регенеративный теплообменник. В этом методе используется трапецоидальное разностное представление уравнений. Метод был запрограмирован для цифровой вычислительной машины, и было проведено сравнение с другим независимым метода расчёта. Необходимо решить вопрос о минимальном числе шагов по пространственной переменной, которые следует использовать при моделировании вычислительной машиной для точного решения уравнений, когда длину шага нельзя легко изменить. Поэтому рассматриваются коэффициенты, учитывающие ошибки вследствие отбрасывания членов. Показано, что чем больше коэффициент Хозена K/K_0, тем меньше нужно шагов для получения точного решения.